The Hodrick-Prescott Filter

Macroeconomics (M8674)

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1. Main goal

Main goal of a filter in macroeconomics

- In economics, any filter intends to separate a time-series y_t into a ***trend*** τ_t and a ***cyclical component*** φ_t such that:

$$y_t = \tau_t + \varphi_t \tag{1}$$

  - The trend is the long-run component of the time series
  - The cyclical component is the short-run component of the time series

- Therefore, from (1) we get

$$\varphi_t = y_t - \tau_t \tag{2}$$

- The trick is to minimize eq. (2) subject to some given constraint on φ_t

Types of Filters

- There are various approaches to separate the ***long-run trend*** from the ***short-run cyclical component*** of a time series $y(t)$.
  - Linear filter
  - Linear filter with breaks
  - Nonlinear filters
- Nonlinear filters
  - Hodrick-Prescott filter (Hodrick & Prescott, 1997)
  - Band Pass filter (Baxter & King, 1999)
  - Hamilton filter (James Hamilton, 2017)
  - … and some others

Don't trust the internet or chatbots

- If you do not know what you are doing, you will make stupid mistakes
- The internet is full of wrong (or incomplete) information
- Even very respected sources may mislead you:
    - Stata may mislead you: here
    - Statsmodels (a famous Python library) may mislead you: here
    - R may mislead you: here
    - Matlab may mislead you: here
- Either you know very well what you are doing, or ...

2. The Hodrick-Prescott filter (HP)

The HP filter: some intuition

- We have ***some data***: a time series y_t
- We want to extract a ***smooth trend*** τ_t from y_t
- We want the ***difference*** between the two (φ_t), to be "acceptable" given what we know about booms and recessions: ***not too large, not too small***
- We introduce a ***parameter*** (λ) into a minimization problem to achieve that
- The ***minimization problem*** with respect to τ_t can be written as:

$$\min_{\tau_t}\left\{\mathcal{L}(\tau)=\sum_{t=1}^{n}\underbrace{\left(y_t-\tau_t\right)^2}_{=\varphi_t^2}+\lambda\sum_{t=2}^{n-1}\underbrace{\left[\left(\tau_{t+1}-\tau_t\right)-\left(\tau_t-\tau_{t-1}\right)\right]^2}_{\text{constraint}}\right\}$$

The HP filter: Special Cases

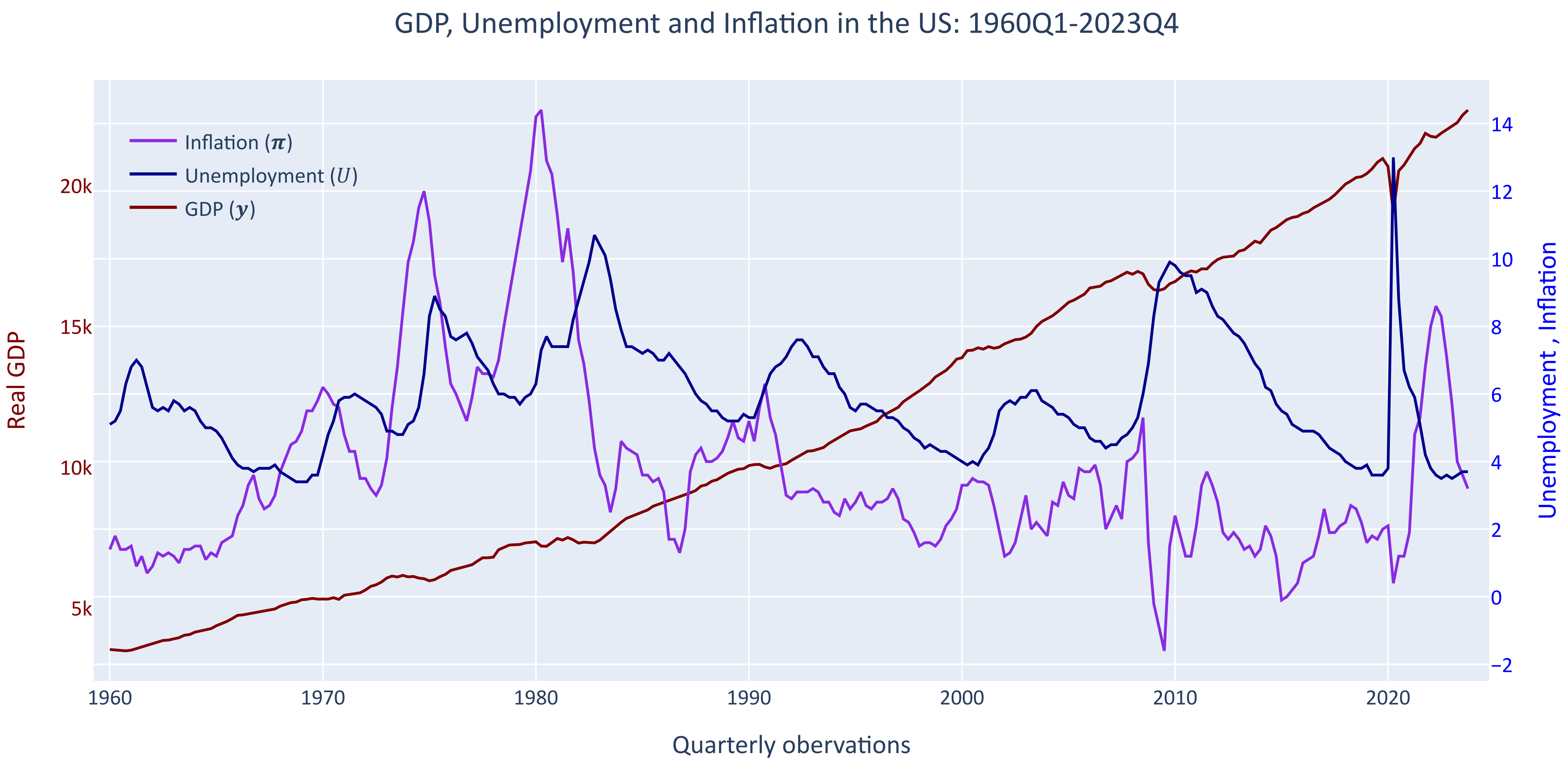
The value given to parameter λ is a choice of ours:

$$\min_{\tau_t} \left\{ \mathcal{L}(\tau) = \sum_{t=1}^{n} (y_t - \tau_t)^2 + \lambda \sum_{t=2}^{n-1} [(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2 \right\}$$

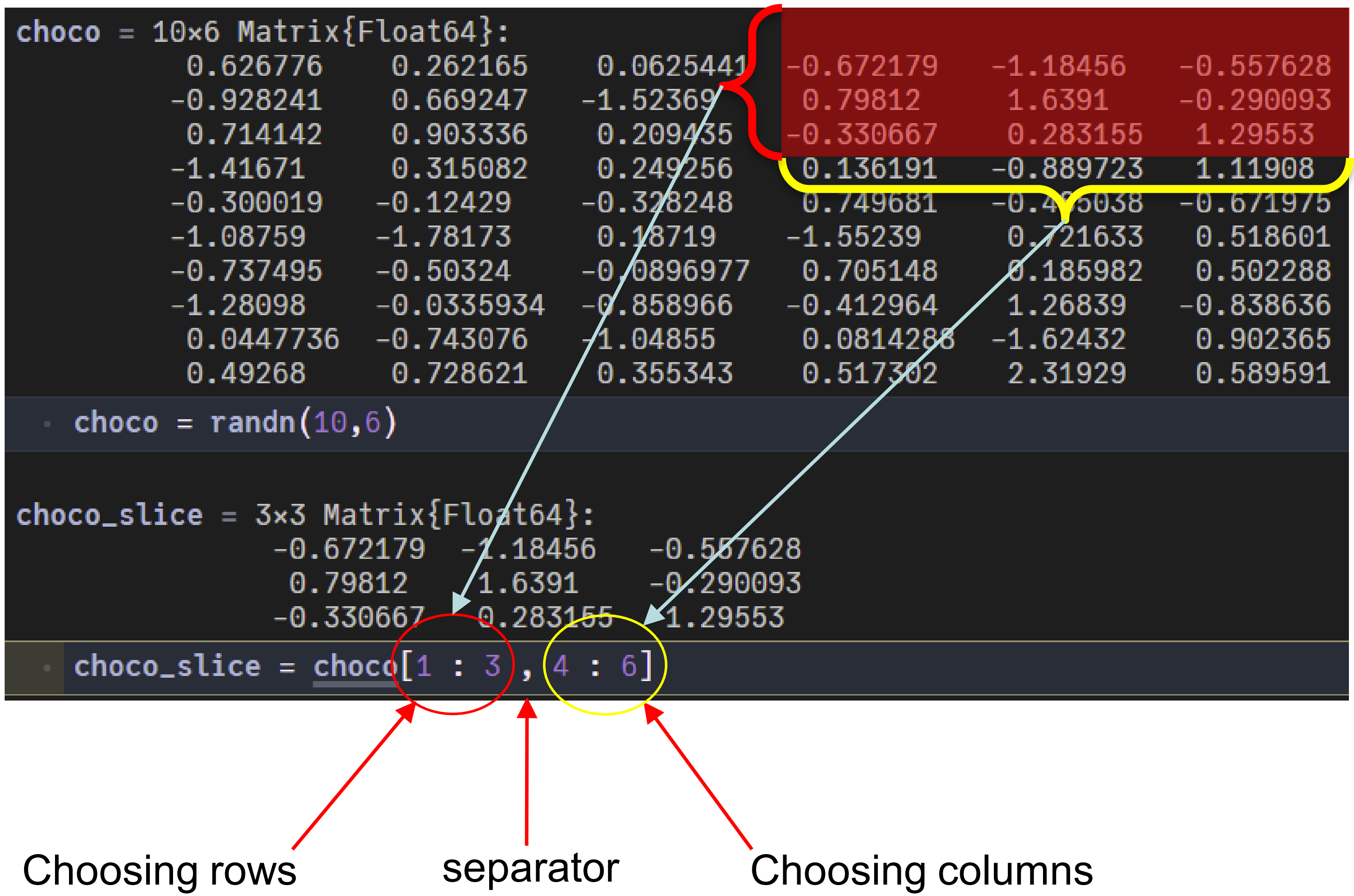
- $\lambda = 0 \Rightarrow$ trivial solution because there are ***no cycles*** : $y_t = \tau_t, \forall t$
- $\lambda \rightarrow \infty \Rightarrow$ linear trend leads to ***huge cycles*** between y_t and τ_t
- $\lambda = 1600 \Rightarrow$ duration/amplitude of cycles acceptable for ***quarterly data***
- $\lambda = 7 \Rightarrow$ duration/amplitude of cycles acceptable for ***annual data***
- There is ***no "unquestionable" value*** for λ

The HP Filter: an Example

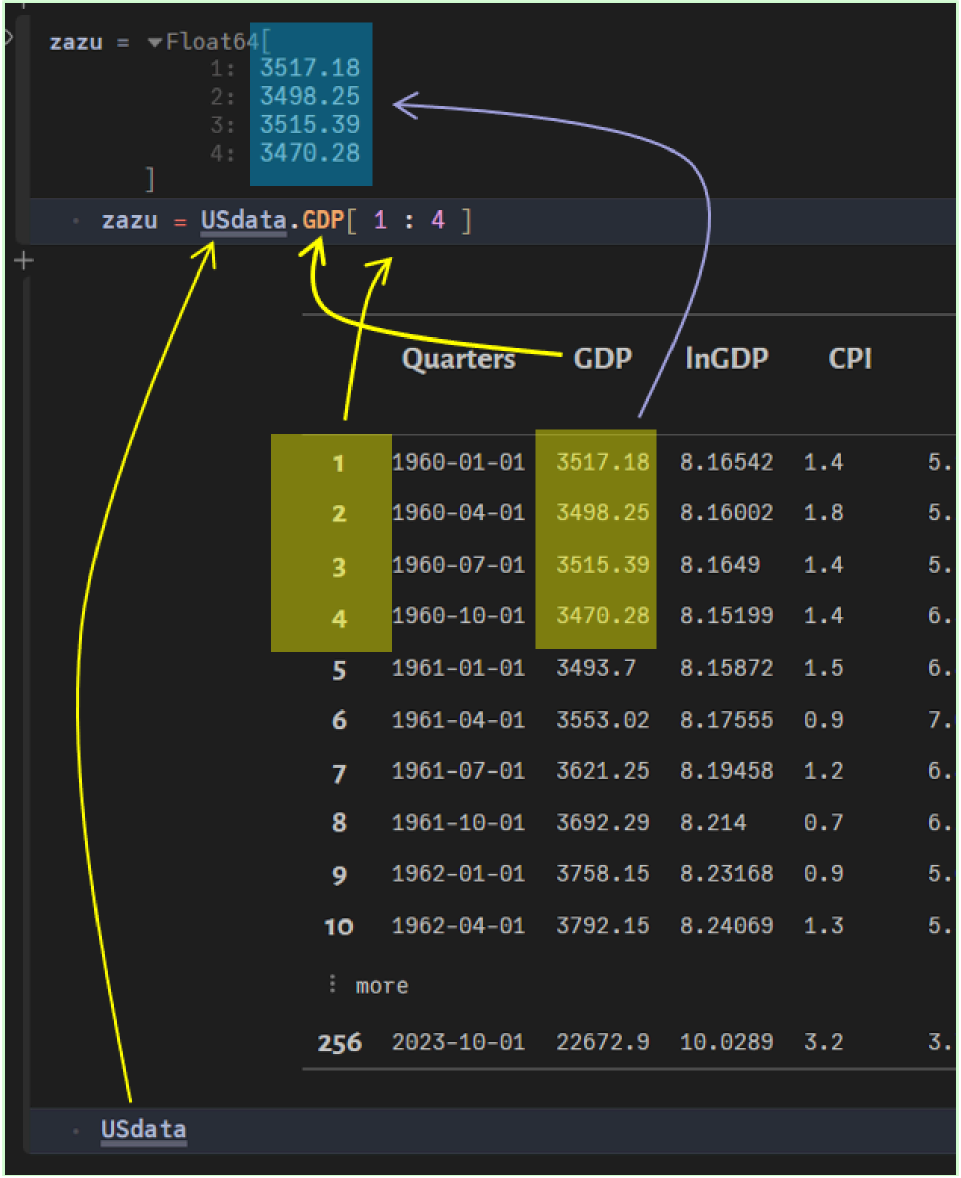
- Main objective: ***obtain cycles as % deviations from the trend***
- This has an important implication:
  - Time series with a trend: ***apply logs*** to the data before extracting the trend and the cycles (`lnGDP`, `lnCPI`)
  - Time series without a clear trend: ***do not apply logs*** to the data (`UR`)
- Quarterly data: `US_data.csv`
- A simple example:
  - Real GDP (column `GDP`)
  - Consumer Price Index (column `CPI`)
  - Unemployment Rate (column `UR`)


GDP, Unemployment and Inflation in the US: 1960Q1-2023Q4
Inflation (π)
Unemployment (U)
GDP (y)
Real GDP
5k
10k
15k
20k
Unemployment , Inflation
−2
0
2
4
6
8
10
12
14
1960
1970
1980
1990
2000
2010
2020
Quarterly obervations

Dealing with rows and columns in a matrix



Dealing with rows and columns in a dataframe



To select a column:

- Use its header (`GDP` in the example)
- It is also possible to use its column number (not shown here)

To select rows:

- Use their numbers (1, 2, 3, ...)
- Range of numbers (1:4 in the example)
- No number, just the header: all rows are selected
- Use a condition (e.g. `USdata[USdata.CPI .< 0 , :]`)

Compute the HP trend and cycles: a single variable

Naming the variable with the trend

Function that computes the trend

Column `lnGDP` in the `USdata`

Smoothing parameter

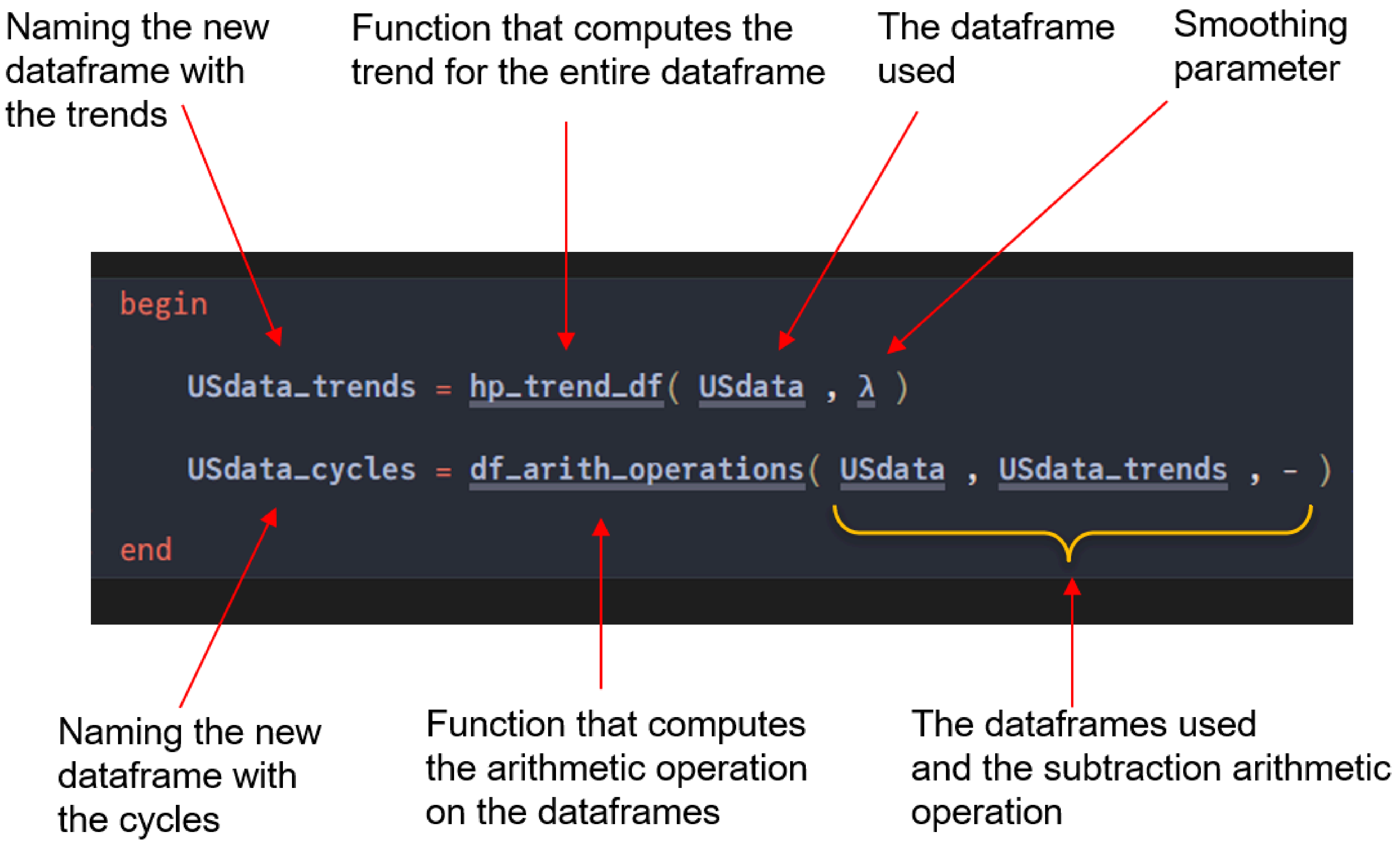
```
begin
    GDP_trend = hp_trend( USdata.lnGDP , λ )      # or #GDP_trend = hp_trend(USdata[:,3], λ)
    GDP_cycles = USdata.lnGDP - GDP_trend          # or GDP_cycles = USdata[:,3] - GDP_trend
end
```

Naming the cycles

Computing the cycles

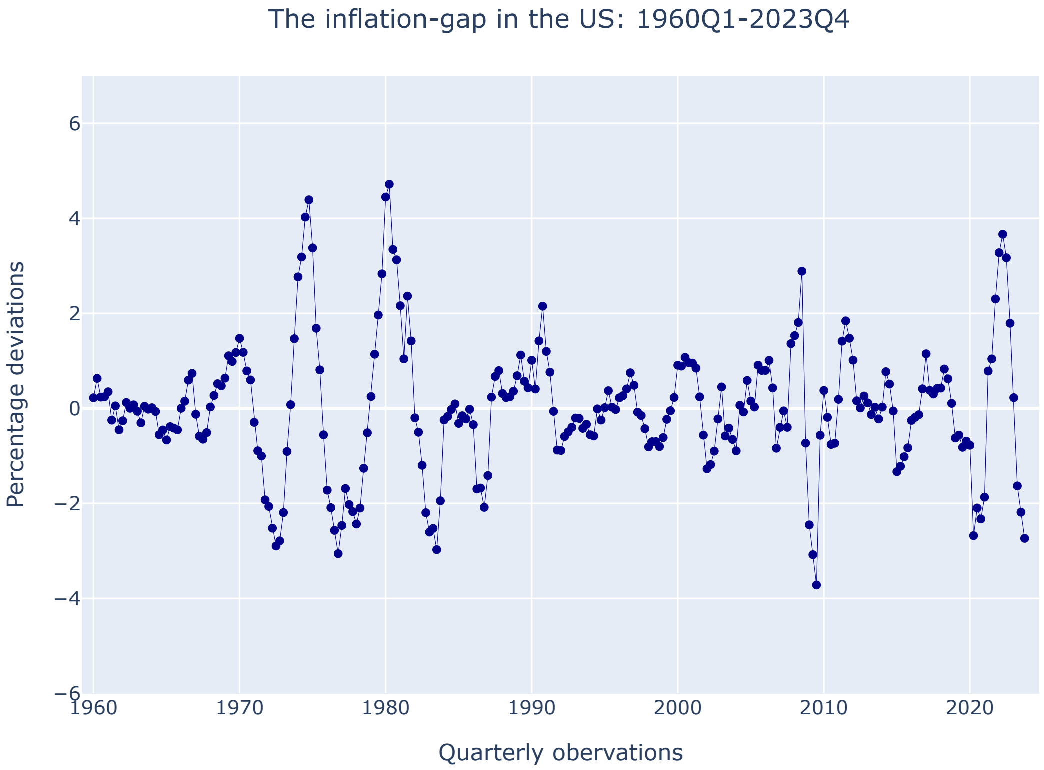
All rows, column 3

Compute the HP trend and cycles: an entire data set

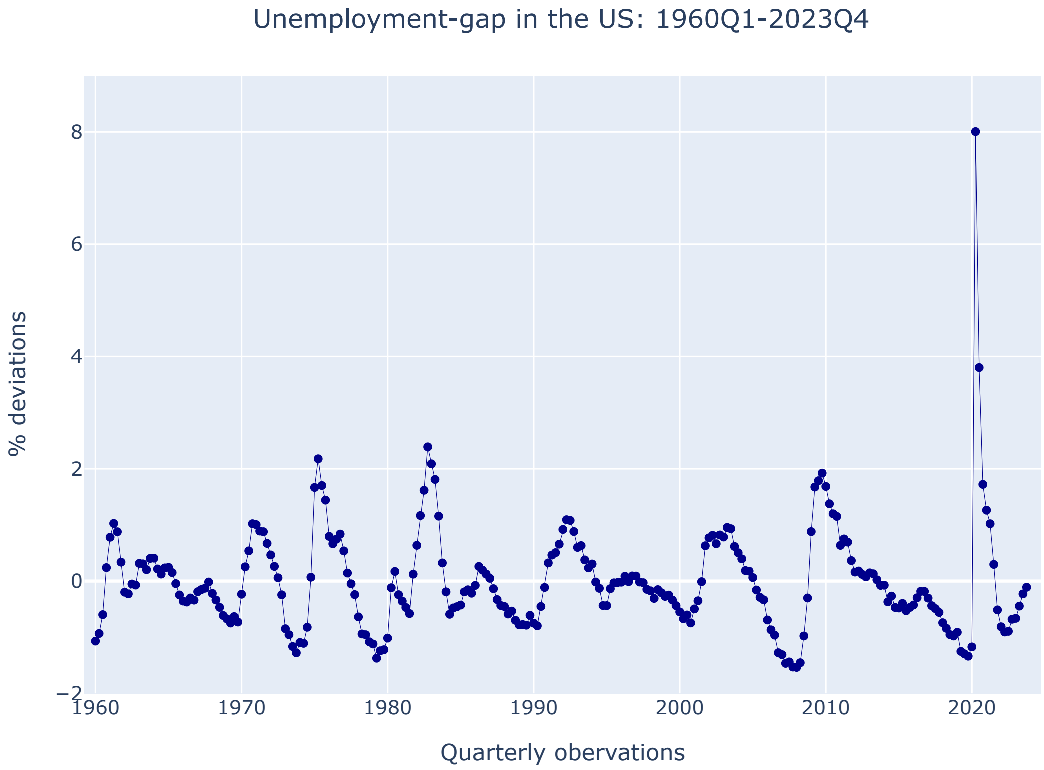


Business cycles: Inflation and Unemployment

The **inflation-gap**

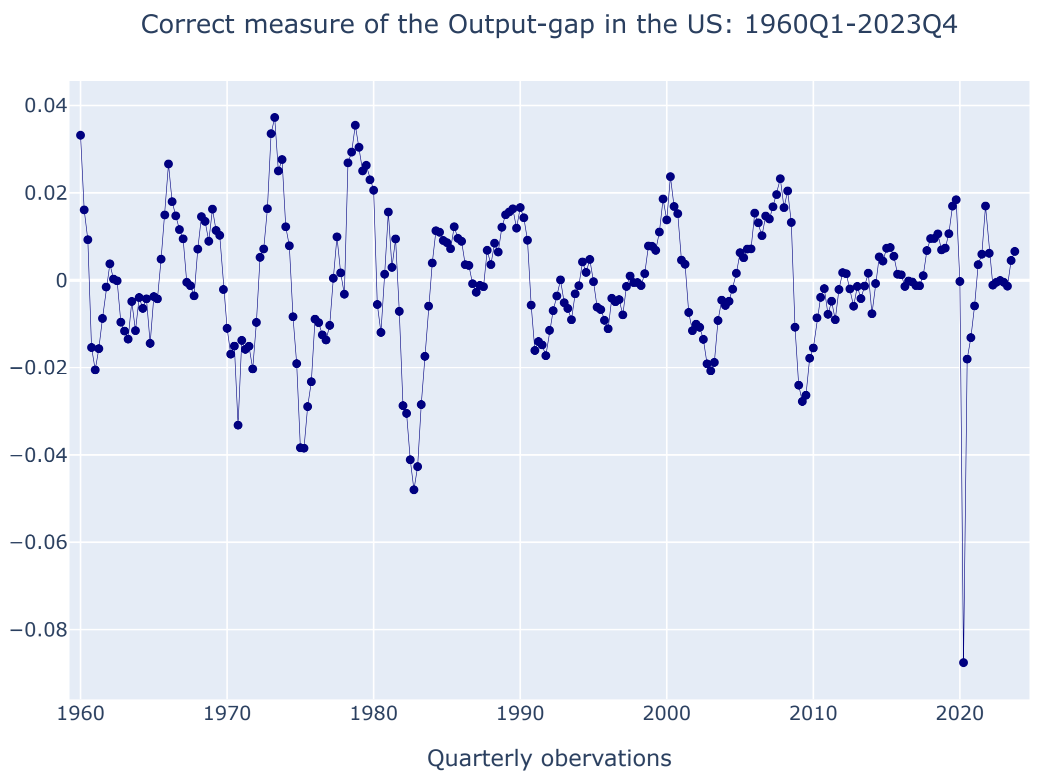


The **unemployment-gap**

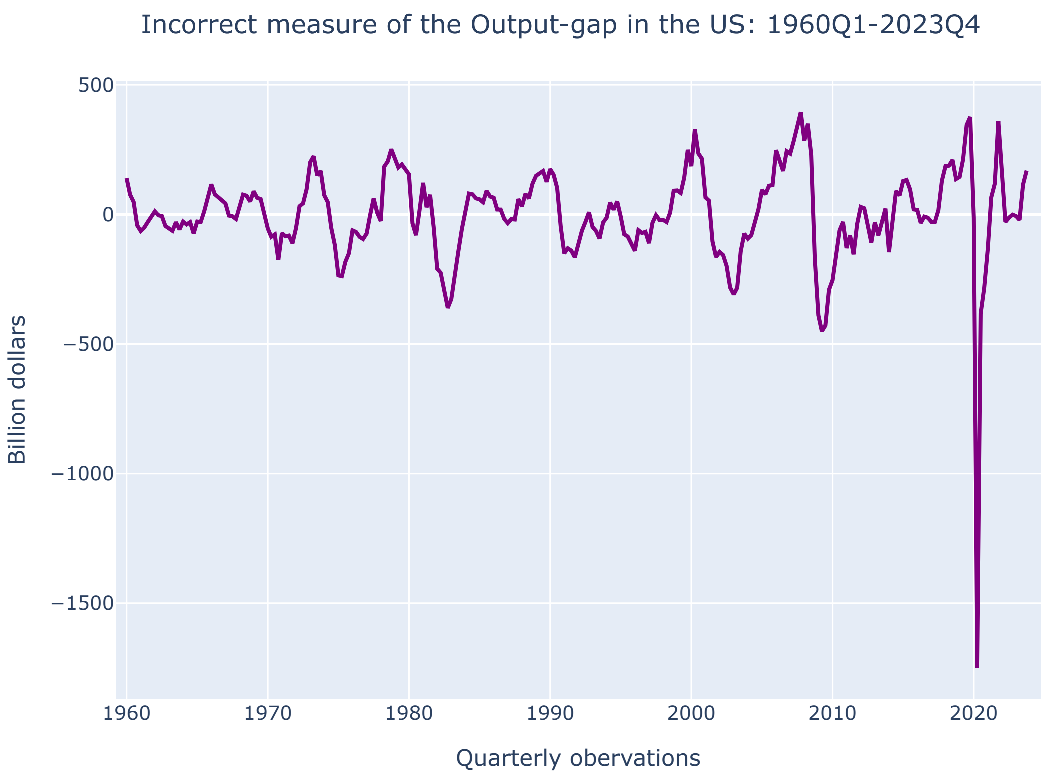


The output-gap: logs vs levels

Correctly measured: using logs



Incorrectly measured: using levels



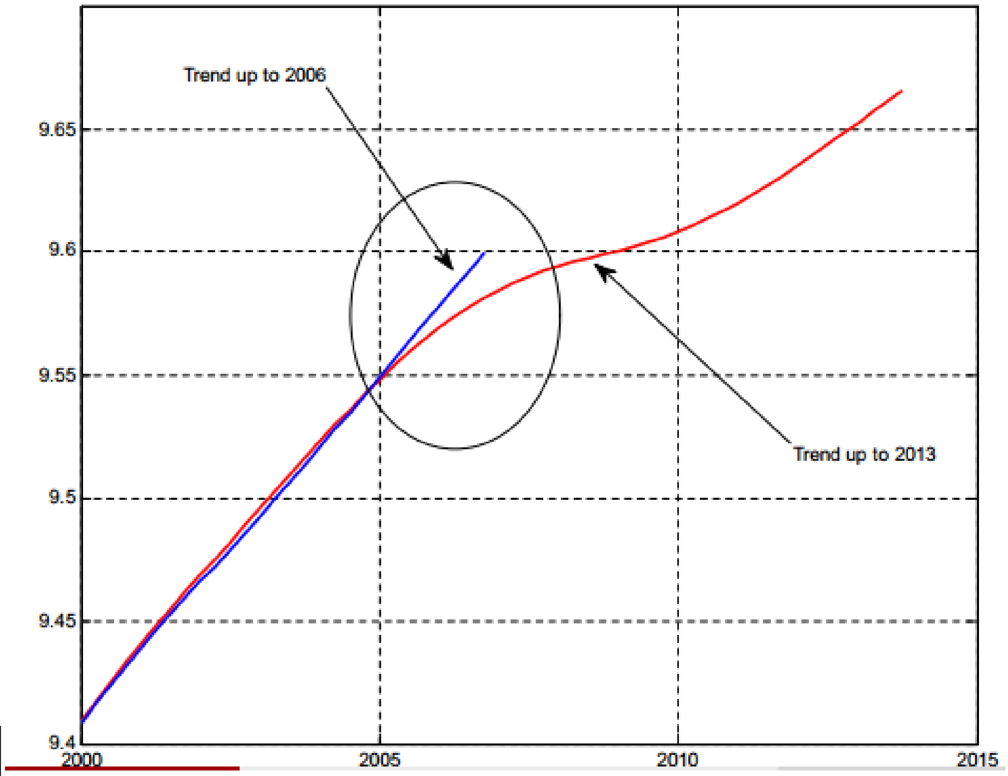
3. Use filters with care

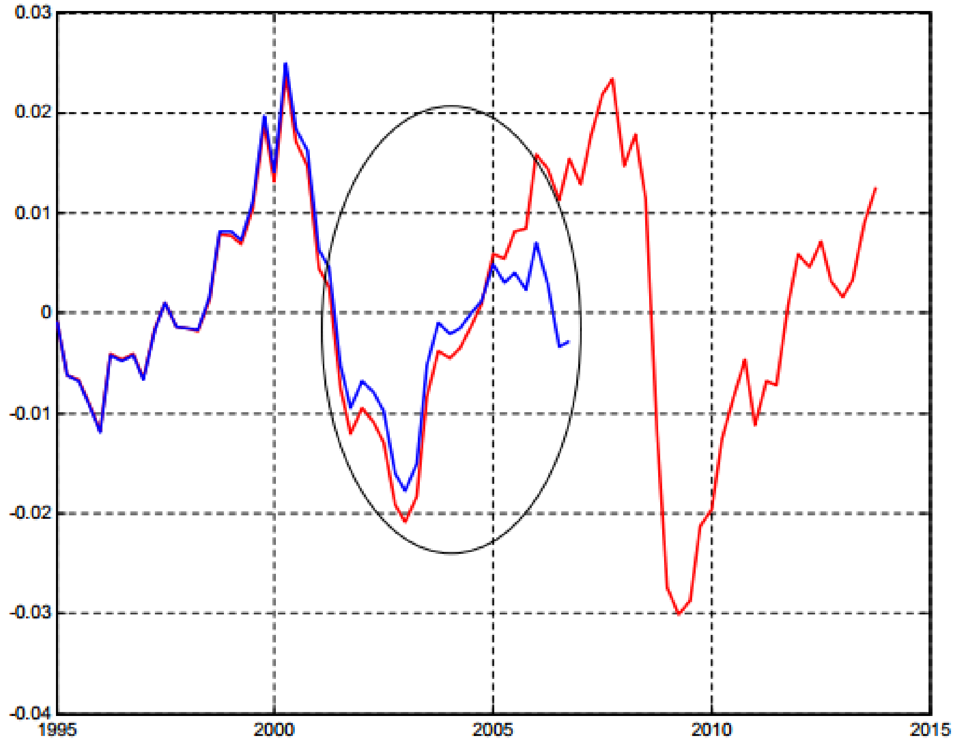
Three Major Issues

- There is ***no perfect filter*** ... but the HP seems very good.
- Measuring Potential GDP and Natural Unemployment ***is difficult***:
  - Potential GDP is usually associated with the HP-trend in GDP ... but not exclusively.
  - The Natural Rate of Unemployment is largely associated with the HP-trend in unemployment.
- The HP filter can be ***misused for policy purposes***:
  - The James Bullard 2012 case in the USA is a well-known example.
- The HP filter is very useful but should be ***used with care***.

Limitations of the HP Filter

- ***New data*** leads to the rewriting of the history of the economy
- The blue lines: data only up to 2008
- The red lines: data up to 2013



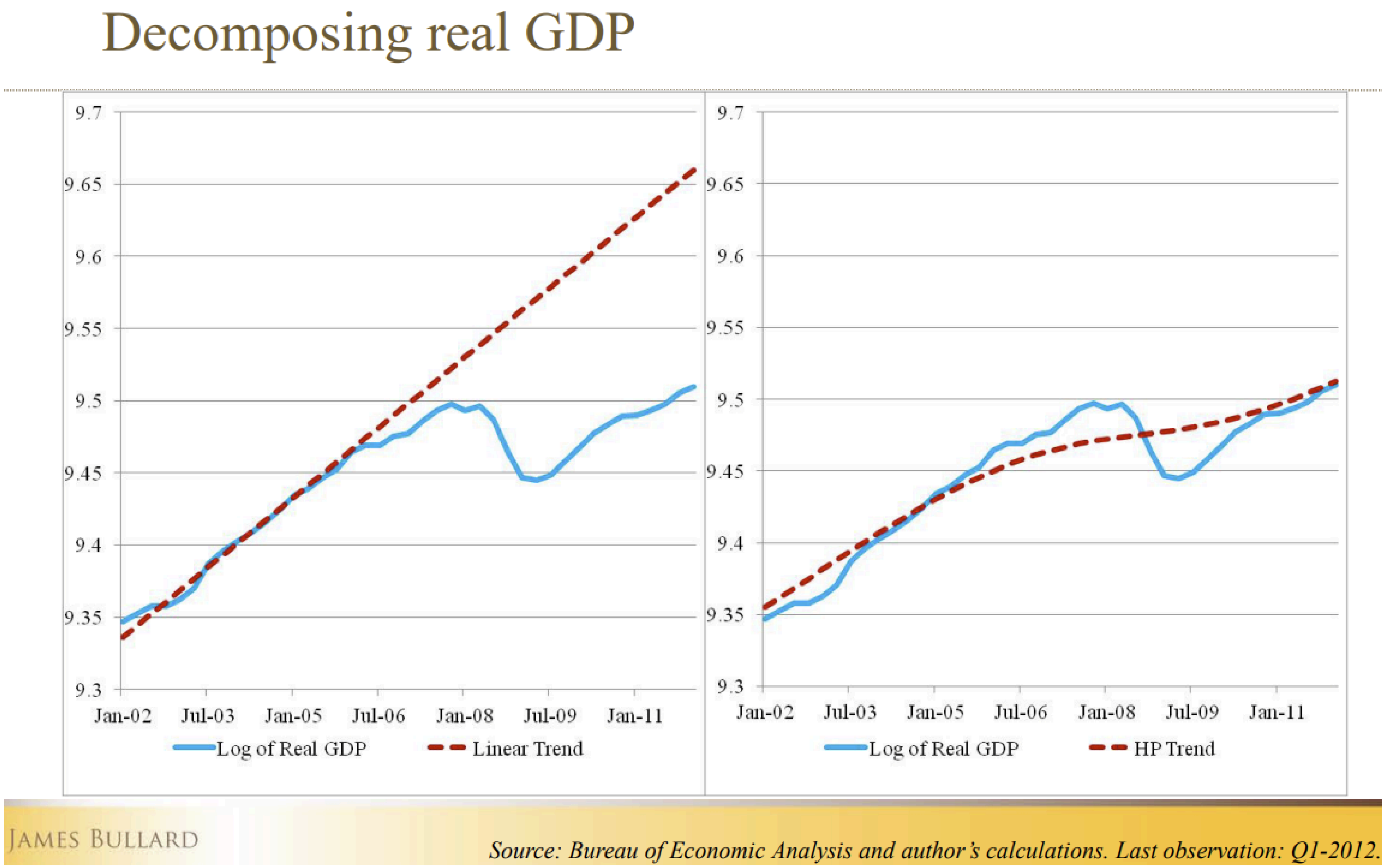


Misuses of the HP Filter

- In 2012, the US economy had an unemployment rate close to 8%, one of the highest rates since WWII.
- The Fed Funds Rate was at 0%, to stimulate the economy.
- The inflation rate was much below the target level (2%) at 0.5% and showing signs of going down.

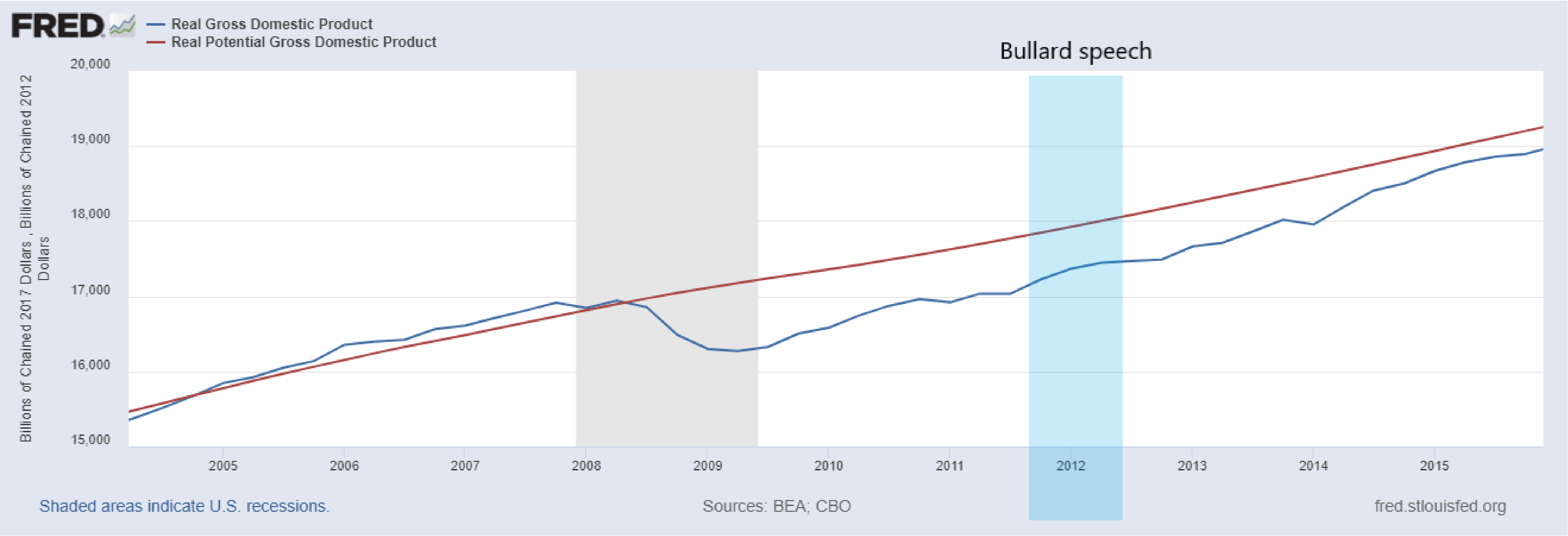
- James Bullard (the President of the Fed of St. Louis), in a famous speech in June 2012 defended that the US economy had gone back to Potential GDP.
  - He strongly pushed for a sharp increase in the Fed Funds Rate.
  - He used the HP-filter to substantiate his proposal.

The HP filter according to James Bullard



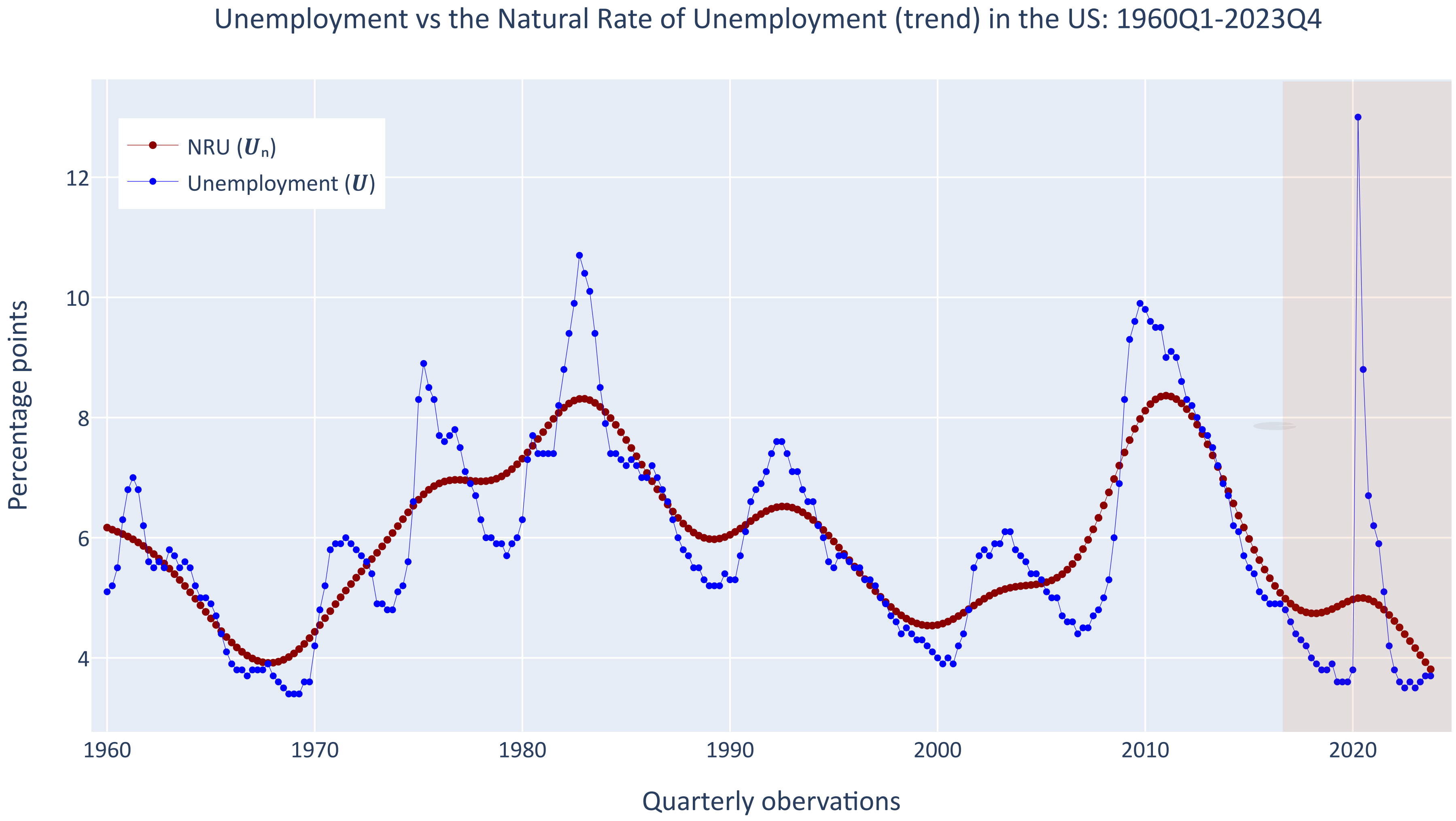
The Output-gap according to the Fed of St. Louis

- The Fed of St. Louis publishes "oficial" US data for Real GDP and Potential GDP.
- Real GDP is the blue line; Potential GDP is the red line.



HP filter & the Natural Unemployment Rate (NUR)

No, Covid-19 did not raise the NUR; no, an increase in NUR did not anticipate Covid-19!



Appendix: HP Filter Derivation

Not compulsory reading

Minimization of the Loss function

The loss function $\mathcal{L}(\tau)$ is given by:

$$\mathcal{L}(\tau) = \sum_{t=1}^{n} (y_t - \tau_t)^2 + \lambda \sum_{t=2}^{n-1} [(\tau_{t+1} - \tau_t) - (\tau_t - \tau_{t-1})]^2$$

To minimize the Loss function with respect to τ_t:

$$\min_{\tau_t} \{\mathcal{L}(\tau)\}$$

Take the derivatives with respect to τ_t and setting them to zero:

$$\frac{\partial \mathcal{L}(\tau)}{\partial \tau_t} = 0, \quad \forall t = 1, \ldots, n$$

Those derivatives are known as the First Order Conditions (FOCs).

First Order Conditions (FOCs)

- For τ_1 :
  - $\mathcal{L}(\tau_1) = (y_1 - \tau_1)^2 + \lambda[\tau_3 - 2\tau_2 + \tau_1)]^2$
  - $\frac{\partial \mathcal{L}(\tau_1)}{\partial \tau_1} = -2\left(y_1 - \tau_1\right) + 2\lambda\left(\tau_3 - 2\tau_2 + \tau_1\right) = 0$
  - $(1+\lambda)\tau_1 - 2\lambda\tau_2 + \lambda\tau_3 = y_1$
- For τ_2 :
  - $\mathcal{L}(\tau_2) = (y_2 - \tau_2)^2 + \lambda[\tau_4 - 2\tau_3 + \tau_2)]^2 + \lambda[(\tau_3 - 2\tau_2 + \tau_1)]^2$
  - $\frac{\partial \mathcal{L}(\tau_2)}{\partial \tau_2} = -2\left(y_2 - \tau_2\right) + 2\lambda\left[\left(\tau_4 - 2\tau_3 + \tau_2\right) - 2\left(\tau_3 - 2\tau_2 + \tau_1\right)\right] = 0$
  - $(1+5\lambda)\tau_2 - 2\lambda\tau_1 - 4\lambda\tau_3 + \lambda\tau_4 = y_2$

First Order Conditions (FOCs): continuation

- For τ_t, such that $t = 3, \ldots, n-2$, the FOC is given by:
  - $-2\left(y_t - \tau_t\right) + 2\lambda\left(\tau_{t-2} - 4\tau_{t-1} + 6\tau_t - 4\tau_{t+1} + \tau_{t+2}\right) = 0$
    or
  - $(1+6\lambda)\tau_t - 4\lambda\tau_{t-1} - 4\lambda\tau_{t+1} + \lambda\tau_{t-2} + \lambda\tau_{t+2} = y_t$
- And for the two right boundary conditions $(n-1, n)$, we have the following FOCs (they are symmetric to those on the left boundary)
  - $(1+5\lambda)\tau_{n-1} - 2\lambda\tau_n - 4\lambda\tau_{n-2} + \lambda\tau_{n-3} = y_{n-1}$
  - $(1+\lambda)\tau_n - 2\lambda\tau_{n-1} + \lambda\tau_{n-2} = y_n$

First Order Conditions (FOCs): Matrix form

The 5 FOCS that represent the optimal conditions on the left boundary, the right boundary, and the interior conditions $(3 \leq t \leq n-2)$ are:

$$
\begin{aligned}
&(1+\lambda)\tau_1 - 2\lambda\tau_2 + \lambda\tau_3 = y_1 \\
&(1+5\lambda)\tau_2 - 2\lambda\tau_1 - 4\lambda\tau_3 + \lambda\tau_4 = y_2 \\
&(1+6\lambda)\tau_t - 4\lambda\left(\tau_{t-1} + \tau_{t+1}\right) + \lambda\left(\tau_{t-2} + \tau_{t+2}\right) = y_t, \quad 3 \leq t \leq n-2 \\
&(1+5\lambda)\tau_{n-1} - 2\lambda\tau_n - 4\lambda\tau_{n-2} + \lambda\tau_{n-3} = y_{n-1} \\
&(1+\lambda)\tau_n - 2\lambda\tau_{n-1} + \lambda\tau_{n-2} = y_n
\end{aligned}
$$

So, we can write:

$$A(\lambda)\tau = y$$

An example with n=6

- $t = 1$

$$\mathcal{L}(\tau_1) = (y_1 - \tau_1)^2 + \lambda(\tau_3 - 2\tau_2 + \tau_1)^2$$

$$\frac{\partial \mathcal{L}}{\partial \tau_1} = -2\left(y_1 - \tau_1\right) + 2\lambda\left(\tau_3 - 2\tau_2 + \tau_1\right) = 0$$

$$(1 + \lambda)\tau_1 - 2\lambda\tau_2 + \lambda\tau_3 = y_1$$

- $t = 2$

$$\mathcal{L}(\tau_2) = (y_2 - \tau_2)^2 + \lambda(\tau_4 - 2\tau_3 + \tau_2)^2 + \lambda(\tau_3 - 2\tau_2 + \tau_1)^2$$

$$\frac{\partial \mathcal{L}}{\partial \tau_2} = -2\left(y_2 - \tau_2\right) + 2\lambda[(\tau_4 - 2\tau_3 + \tau_2) - 2\left(\tau_3 - 2\tau_2 + \tau_1\right)] = 0$$

$$(1 + 5\lambda)\tau_2 - 2\lambda\tau_1 - 4\lambda\tau_3 + \lambda\tau_4 = y_2$$

An example with n=6 (cont.)

- $t = 3$

$$\mathcal{L}(\tau_3) = (y_3 - \tau_3)^2 + \lambda(\tau_5 - 2\tau_4 + \tau_3)^2 + \lambda(\tau_4 - 2\tau_3 + \tau_2)^2 + \lambda(\tau_3 - 2\tau_2 + \tau_1)^2$$

$$\frac{\partial \mathcal{L}}{\partial \tau_3} = -2\left(y_3 - \tau_3\right) + 2\lambda\left[(\tau_5 - 2\tau_4 + \tau_3) - 2\left(\tau_4 - 2\tau_3 + \tau_2\right) + \left(\tau_3 - 2\tau_2 + \tau_1\right)\right]$$

$$(1 + 6\lambda)\tau_3 - 4\lambda\tau_2 - 4\lambda\tau_4 + \lambda\tau_1 + \lambda\tau_5 = y_3$$

- $t = 4$

$$\mathcal{L}\left(\tau_4\right) = (y_4 - \tau_4)^2 + \lambda(\tau_6 - 2\tau_5 + \tau_4)^2 + \lambda(\tau_5 - 2\tau_4 + \tau_3)^2 + \lambda(\tau_4 - 2\tau_3 + \tau_2)^2 + \lambda(\tau_3 - 2\tau_2 + \tau_1)^2$$

$$\frac{\partial \mathcal{L}}{\partial \tau_4} = -2\left(y_4 - \tau_4\right) = 2\lambda[(\tau_6 - 2\tau_5 + \tau_4) - 2\left(\tau_5 - 2\tau_4 + \tau_3\right) + \left(\tau_4 - 2\tau_3 + \tau_2\right)] = 0$$

$$(1 + 6\lambda)\tau_4 - 4\lambda\tau_3 - 4\lambda\tau_5 + \lambda\tau_2 + \lambda\tau_6 = y_4$$

An example with n=6 (cont.)

- $t = 5$

$$\mathcal{L}(\tau_5) = (y_5 - \tau_5)^2 + \lambda(\tau_7 - 2\tau_6 + \tau_5)^2 + \lambda(\tau_6 - 2\tau_5 + \tau_4)^2 + \ldots$$

- This is not a viable way to writing down the Lagrangian function to minimize the problem because τ_7 is not an interior index, as we have only 6 observations.
- We have to apply the boundary condition that the number of observations end at $t = 6$.
- Then we will move backward to $t = 5$.
- When we get the results for τ_6 and τ_5, the entire solution set is known.

An example with n=6 (cont.)

At the initial boundary we iterate forward, at the end boundary we iterate backwards

- $t = 6$

$$\mathcal{L}(\tau_6) = (y_6 - \tau_6)^2 + \lambda(\tau_6 - 2\tau_5 + \tau_4)^2$$

$$\frac{\partial \mathcal{L}}{\partial \tau_6} = -2\left(y_6 - \tau_6\right) + 2\lambda\left(\tau_6 - 2\tau_5 + \tau_4\right) = 0$$

$$(1 + \lambda)\tau_6 - 2\lambda\tau_5 + \lambda\tau_4 = y_6$$

- $t = 5$

$$\mathcal{L}(\tau_5) = (y_5 - \tau_5)^2 + \lambda(\tau_5 - 2\tau_4 + \tau_3)^2 + \lambda(\tau_6 - 2\tau_5 + \tau_4)^2$$

$$\frac{\partial \mathcal{L}}{\partial \tau_5} = -2\left(y_5 - \tau_5\right) + 2\lambda\left[\left(\tau_5 - 2\tau_4 + \tau_3\right) - 2\left(\tau_6 - 2\tau_5 + \tau_4\right)\right]$$

$$(1 + 5\lambda)\tau_5 - 4\lambda\tau_4 - 2\lambda\tau_6 + \lambda\tau_3 = y_5$$

An example with n=6: all FOCs

$$t = 1 \rightarrow (1+\lambda)\tau_1 - 2\lambda\tau_2 + \lambda\tau_3 = y_1$$
$$t = 2 \rightarrow (1+5\lambda)\tau_2 - 2\lambda\tau_1 - 4\lambda\tau_3 + \lambda\tau_4 = y_2$$
$$t = 3 \rightarrow (1+6\lambda)\tau_3 - 4\lambda\tau_2 - 4\lambda\tau_4 + \lambda\tau_1 + \lambda\tau_5 = y_3$$
$$t = 4 \rightarrow (1+6\lambda)\tau_4 - 4\lambda\tau_3 - 4\lambda\tau_5 + \lambda\tau_2 + \lambda\tau_6 = y_4$$
$$t = 5 \rightarrow (1+5\lambda)\tau_5 - 4\lambda\tau_4 - 2\lambda\tau_6 + \lambda\tau_3 = y_5$$
$$t = 6 \rightarrow (1+\lambda)\tau_6 - 2\lambda\tau_5 + \lambda\tau_4 = y_6$$

An example with n=6

- If n = 6, the dense form of A is given by:

$$A = \begin{bmatrix} 1+\lambda & -2\lambda & \lambda & 0 & 0 & 0 \\ -2\lambda & 1+5\lambda & -4\lambda & \lambda & 0 & 0 \\ \lambda & -4\lambda & 1+6\lambda & -4\lambda & \lambda & 0 \\ 0 & \lambda & -4\lambda & 1+6\lambda & -4\lambda & \lambda \\ 0 & 0 & \lambda & -4\lambda & 1+5\lambda & -2\lambda \\ 0 & 0 & 0 & \lambda & -2\lambda & 1+\lambda \end{bmatrix}$$

- **Notice the symmetry of matrix A**
  - $diag_2 = [\lambda, \lambda, \lambda, \lambda]$.
  - $diag_1 = [-2\lambda, -4\lambda, -4\lambda, -4\lambda, -2\lambda]$,
  - $diag_0 = [1+\lambda, 1+5\lambda, 1+6\lambda, 1+6\lambda, 1+5\lambda, 1+\lambda]$,

Notebook implementation

- Matrix A non-trivial diagonals:
  - $diag_2 = [\lambda, \lambda, \lambda, \lambda]$.
  - $diag_1 = [-2\lambda, -4\lambda, -4\lambda, -4\lambda, -2\lambda]$,
  - $diag_0 = [1+\lambda, 1+5\lambda, 1+6\lambda, 1+6\lambda, 1+5\lambda, 1+\lambda]$,
  - $diag_{-1} = [-2\lambda, -4\lambda, -4\lambda, -4\lambda, -2\lambda]$,
  - $diag_{-2} = [\lambda, \lambda, \lambda, \lambda]$.
- Check the function `hp_trend` in the notebook `HP_IRF_2026.jl`

4. Readings

Point 2

- For this point, there is no compulsory reading.
- However, Dirk Krueger (2007). "Quantitative Macroeconomics: An Introduction" (Chapter 2), manuscript, Department of Economics University of Pennsylvania, is well suited for the material covered here.
- This text is a small one (12 pages), easy to read, and beneficial for studying the stylized facts of business cycles, mainly to understand how the Hodrick-Prescott filter is calculated. However, notice that, as mentioned, it is not compulsory reading.

Point 3

- No textbook covers the topics/controversies mentioned in this section.
- This coursework intends to provide a framework for a better understanding of these controversies at the end of the course.
- All we have to handle is:
  - A little bit of mathematics
  - A little bit of computation
  - A little bit of macroeconomics